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# CONVECTIVE WALL PLUME: HIGHER ORDER ANALYSIS

NOOR AFZAL
Department of Mechanical Engineering, Aligarh Muslim University, Aligarh, India

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**Abstract**—A study of natural convection flow arising from a steady line thermal source embedded at the leading edge of a vertical surface is carried out for moderately large values of Grashof number by the method of matched asymptotic expansions. The first and second order boundary-layer equations are studied when the medium is bounded by an infinite horizontal surface placed at an arbitrary distance below the convective wall plume. The numerical solutions are obtained for two fluids, namely air and water. It is shown that the structure of the convected wall plume depends strongly on the location of horizontal plane. Separate results for prescribed wall temperature and for an adiabatic convective wall plume are presented.

## NOMENCLATURE

- $d$, nondimensional distance of horizontal plane from the source;
- $c_f$, coefficient of skin friction, $2\tau_w/\rho U^2$;
- $f_1, f_2$, first and second order stream functions defined by (18a) and (36);
- $g$, gravitational acceleration;
- $g_1, g_2$, first and second order temperatures defined by (18a) and (36);
- $Gr$, Grashof number, $g\beta T_r L^3/\nu^2$;
- $Gr_x$, local Grashof number, $g\beta T_r x^3/\nu^2$;
- $J$, nondimensional global heat flux defined by (6);
- $J_1, J_2$, first and second order global heat flux defined by (11);
- $K$, thermal conductivity of the fluid;
- $L$, reference length;
- $N_Q$, global Nusselt number, $Q/K(T_w - T_\infty)$;
- $Nu_x$, local Nusselt number, $xq_w(x)/K(T_w - T_\infty)$;
- $q_w$, conductive heat flux from wall plume;
- $q_s$, $Q_s$, nondimensional and dimensional heat input by the thermal source;
- $Q$, global heat flux from convective wall plume;
- $r$, recovery factor, $(T_a - T_\infty)/T_r$;
- $T$, temperature;
- $T_a$, adiabatic wall temperature;
- $T_r$, reference temperature, $Q_s/KJ_1$;
- $T_w$, prescribed wall temperature, $T_\infty + T_r Gr_x^{-1/5}$;
- $U$, characteristic free convection speed, $\nu Gr^{2/5}/L$;
- $x_c$, value of $x$ where displacement speed vanishes;
- $x$, $y$, coordinates along the wall plume measured from heat source and normal to it;
- $Y$, boundary-layer variable, $yGr^{1/5}$.

Greek symbols

- $\beta$, volumetric thermal expansion coefficient;
- $\varepsilon$, local perturbation parameter, $Gr_x^{-1/5}$;
- $\varepsilon_1$, reference perturbation parameter, $Gr^{-1/5}$
- $\phi$, nondimensional temperature, $(T - T_\infty)Gr^{1/5}/T_r$;
- $\phi_1, \phi_2$, first and second order temperatures in inner region;
- $\lambda$, unspecified eigen constant;
- $\rho$, fluid density;
- $\sigma$, Prandtl number;
- $\tau_w$, wall shear;
- $\psi$, nondimensional stream function;
- $\psi_1, \psi_2$, first and second order stream function in the inner region;
- $\Psi_1$, $\Psi_2$, first and second order stream function in the outer region;
- $\nu$, kinematic viscosity.

## 1. INTRODUCTION

THE FLOW induced by buoyancy due to a horizontal line source embedded at the leading edge of vertical plate in a fluid which is otherwise at rest is studied for moderately large values of Grashof numbers by the method of matched asymptotic expansions. The problem is of interest in numerous engineering applications, e.g. electronic circuitry where electronic devices mounted on a vertical surface dissipate energy at constant rate, production processes where selective and local heating gives rise to a constant thermal input over a surface. The similarity solutions for the classical laminar boundary layer equations for a convective adiabatic wall plume have been studied by Zimin and Lyakhov [1] and Jaluria and Gebhart [2]. Measurements for turbulent wall plume have been reported by Grella and Faeth [3].

The overall objectives of the present work on convective wall plume are three-fold. Firstly, to analyse the Navier–Stokes equations for moderately large values of Grashof number by the method of matched asymptotic expansions. In particular the second order corrections to [1] and [2] for moderately large Grashof numbers have been predicted. Secondly, the effects of restricted domain, when a horizontal infinite plane boundary is introduced at an arbitrary distance below the convected wall plume, have been investigated. It is shown that outer layer entrainment and the boundary-layer flow pattern is considerably different as the buoyancy layer can no longer entrain the fluid from all parts of an infinite domain. Thirdly, to study the global heat transfer through the integration of energy equation for a large control surface enclosing the leading edge of the plate embedding the source following a procedure analogous to Imai [4].

The results for the two cases of adiabatic wall and prescribed wall temperature of convective wall plume are presented separately. For the adiabatic wall case the adiabatic wall temperature and recovery factor are determined and for the case of prescribed wall temperature the local and global heat transfer are estimated.

## 2. GOVERNING EQUATIONS

Let a horizontal line thermal source of heat, with a steady input $Q_s$, embedded at the leading edge of vertical plate, be placed in an extensive unstratified fluid of uniform temperature. The vorticity transport and energy equations in nondimensional form under usual Boussinesq approximation are

$$\left(\psi_y\frac{\partial}{\partial x}-\psi_x\frac{\partial}{\partial y}\right)\nabla^2\psi+Gr^{-2/5}\nabla^4\psi+\frac{\partial\phi}{\partial y}=0, \quad (1)$$

$$\left(\psi_y\frac{\partial}{\partial x}-\psi_x\frac{\partial}{\partial y}\right)\phi+Gr^{-2/5}\sigma^{-1}\nabla^2\phi=0. \quad (2)$$

The no slip boundary condition on the vertical plate is

$$y=0,\ \psi=\psi_y=0 \quad (3)$$

and that of the prescribed wall temperature or the adiabatic wall is

$$y=0,\ \phi=(T_w-T_\infty)Gr^{1/5}/T_r \quad \text{or} \quad \frac{\partial\phi}{\partial y}=0. \quad (4)$$

Far away from the plate the ambient conditions are

$$x^2+y^2\to\infty,\quad \psi_y\to 0,\quad \phi\to 0. \quad (5)$$

The global heat flux $Q$ is given by (see Appendix)

$$Q(x)=KT_rJ,\ J=Gr^{1/5}\left[\sigma\int_0^\infty\psi_y\phi\,\mathrm{d}y-Gr^{-2/5}\times\int_0^\infty\phi_x\mathrm{d}y+O(Gr^{-2/5})\right]. \quad (6)$$

## 3. ASYMPTOTIC ANALYSIS

The asymptotic solutions of the problem formulated in section 2 are studied for large values of Grashof numbers by the method of matched asymptotic expansions. We seek two limits and two corresponding asymptotic expansions which describe the flow region close to the wall and away from it. The length scales for these flow regions are of order $Gr^{-1/5}$ and unity. The outer limit is defined as $x$ and $y$ fixed as $Gr\to\infty$ and the outer layer, essentially inviscid, can be studied in the term of outer expansions

$$\psi=\Psi_1(x,y)+\varepsilon_1\Psi_2(x,y)+O(\varepsilon_2)$$
$$\phi=\Phi_1(x,y)+\varepsilon_1\Phi_2(x,y)+O(\varepsilon_2) \quad (7)$$

where $\varepsilon_1$ and $\varepsilon_2$ are the gauge functions, to be determined. The inner region close to the wall, where diffusive effects are important, is characterized by buoyancy forces. An order of magnitude analysis leads to the following inner variable

$$Y=yGr^{1/5} \quad (8)$$

and the inner limit is $x$, $Y$ fixed for $Gr\to\infty$. The inner asymptotic expansions are

$$\psi=Gr^{-1/5}[\psi_1(x,Y)+\Delta_1\psi_2(x,Y)+O(\Delta_2)]$$
$$\phi=\phi_1(x,Y)+\Delta_1\phi_2(x,Y)+O(\Delta_2). \quad (9)$$

where $\Delta_1$ and $\Delta_2$ are the gauge functions. The matching of the outer and inner expansions in the overlap region leads to

$$\varepsilon_1=\Delta_1=Gr^{-1/5}. \quad (10)$$

The expansion for global heat flux may also be written as

$$J=J_1+\varepsilon_1J_2+O(\varepsilon_1). \quad (11)$$

(i) *First order problem*

The solution to the leading terms in outer expansions (7) satisfying the outer boundary condition (5)

$$\Psi_1(x,y)=\Phi_1(x,y)=0 \quad (12)$$

shows that there is no flow or temperature induced in the outer layer, when Grashof numbers are large. The trivial solution fails to explain the convection of heat released by the source. The nonuniformity, near the plate, can be analysed by inner expansions (9). Its leading terms satisfy the following equations:

$$\psi_{1Y}\psi_{1xY}-\psi_{1x}\psi_{1YY}+\psi_{1YYY}+\phi_1=0 \quad (13)$$

$$\psi_{1Y}\phi_{1x}-\psi_{1x}\phi_{1Y}+\sigma^{-1}\phi_{1YY}=0. \quad (14)$$

Equation (13) has been written after integrating it once with respect to $Y$ and the resulting function of $x$ set equal to zero. The boundary conditions and the matching conditions are

$$Y=0,\quad \psi_1=\psi_{1Y}=0,\quad \phi_1=(T_1-T_\infty)Gr^{1/5}/T_r \quad \text{or} \quad \phi_{1Y}=0 \quad (15)$$

$$Y\to\infty,\ \psi_{1Y}\to 0,\ \phi_1\to 0. \quad (16)$$

The global heat flux condition gives

$$J_1=\sigma\int_0^\infty\psi_{1Y}\phi_1\,\mathrm{d}Y, \quad (17)$$

and the heat input from the line source $Q_s=KT_rJ_1$. The similarity variables, that are slightly different from [1] and [2],

$$\psi_1 = x^{3/5} f_1(\eta),\ \phi_1 = x^{-3/5} g_1(\eta),\ \eta = Y/x^{2/5}, \quad (18)$$

imply that the temperature of wall is assumed to depend upon $x$ as

$$T_w(x) = T_\infty + Gr^{-1/5} T_r x^{-3/5}. \quad (19)$$

Introducing the similarity variables in equations (13)–(17) we get

$$f_1''' + \tfrac{3}{5} f_1 f_1'' - \tfrac{1}{5} f_1'^2 + g_1 = 0, \quad (20)$$

$$g_1'' + \tfrac{3}{5}\sigma(f_1 g_1)' = 0, \quad (21)$$

$$f_1(0) = f_1'(0) = 0, \quad g_1(0) = 1 \quad \text{or } g_1'(0) = 0, \quad (22a, b, c)$$

$$f_1'(\infty) = g_1(\infty) = 0, \quad (23a, b)$$

$$J_1 = \sigma \int_0^\infty f_1' g_1 \, d\eta. \quad (24)$$

The equations are identical with two dimensional buoyant plume [5, 6] but the boundary conditions (22b) and (22c) are different. An integration of (21) along with (23a) gives

$$g_1' + \tfrac{3}{5}\sigma f_1 g_1 = 0. \quad (25a)$$

Using the boundary condition (22a) we get

$$g_1'(0) = 0 \quad (25b)$$

implying that, to the lowest order, the vertical plate is always adiabatic. However, our study of second order effects for prescribed wall temperature predicts finite heat transfer and thus cannot be regarded as adiabatic. The matching of the leading terms for stream function in inner and outer expansions (7) and (9) leads to

$$\left.\frac{\partial \Psi_1}{\partial x}\right|_{y=0} = \left.\frac{\partial \psi_1}{\partial x}\right|_{Y=0} \equiv \tfrac{3}{5} f_1(\infty) x^{-2/5}, \quad (26)$$

the required boundary condition for the second order outer problem. This shows that outer flow is sink-like in character.

(ii) *Second order problem*

In the outer layer, the solution of the second order term for temperature is $\Phi_2(x, y) = 0$ and the vorticity equation leads to an irrotational type flow governed by

$$\nabla^2 \Psi_2 = 0. \quad (27)$$

The boundary condition (26) provided by matching and the requirement of flow symmetry are

$$\Psi_2(x, y = 0) = f_1(\infty) x^{3/5}, \ x \geq 0 \quad (28a)$$

$$= 0 \qquad , \ x < 0. \quad (28b)$$

The solution of the problem is

$$\Psi_2 = f_1(\infty)(x^2 + y^2)^{3/10} \times \sin\left[\frac{3}{5}\left(\pi - \tan^{-1}\frac{y}{x}\right)\right] \Big/ \sin\frac{3\pi}{5} \quad (29)$$

The matching of second order tangential components of velocity in the expansions demands

$$\Psi_{2y}(x, 0) = \psi_{2y}(x, \infty) = A_1 x^{-2/5} \quad (30)$$

$$A_1 = \tfrac{3}{5} f_1(\infty) \cot\left(\frac{2\pi}{5}\right). \quad (31)$$

The second order terms in inner expansions (9) give the second order equations in the inner region. Integrating the first equation with respect to $Y$ and setting the resulting function of integration to zero we get

$$\psi_{1Y}\psi_{2xY} - \psi_{1x}\psi_{2YY} + \psi_{2Y}\psi_{1xY} - \psi_{2x}\psi_{1YY} + \psi_{2YYY} + \phi_2 = 0 \quad (32)$$

$$\psi_{1Y}\phi_{2x} - \psi_{1x}\phi_{2Y} + \psi_{2Y}\phi_{1x} - \psi_{2x}\phi_{1Y} + \sigma^{-1}\phi_{2YY} = 0 \quad (33)$$

The boundary, matching and global heat flux conditions are

$$Y = 0,\ \psi_2 = \psi_{2Y} = 0,\ \phi_2 = 0, \quad \text{or } \phi_2' = 0, \quad (34)$$

$$Y \to \infty,\ \psi_{2Y} \to A_1 x^{-2/5},\ \phi_2 \to 0 \quad (35)$$

$$J_2 = \sigma \int_0^\infty \psi_{2Y}\phi_1 + \psi_{1Y}\phi_2 \, dY. \quad (36)$$

Introducing the similarity variables

$$\psi_2 = f_2(\eta),\ \phi_2 = x^{-6/5} g_2(\eta) \quad (37)$$

the second order problem (32)–(36) reduces to

$$f_2''' + \tfrac{3}{5} f_1 f_2'' + \tfrac{1}{5} f_1' f_2' + g_2 = 0 \quad (38)$$

$$g_2'' + \tfrac{3}{5}\sigma(f_1 g_2' + 2 f_1' g_2 + f_2' g_1) = 0 \quad (39)$$

$$f_2(0) = f_2'(0) = 0,\ g_2(0) = 0 \quad \text{or } g_2'(0) = 0 \quad (40)$$

$$f_2'(\infty) = A_1,\ g_2(\infty) = 0 \quad (41a, b)$$

and the integration of (39) leads to

$$J_2 = \sigma \int_0^\infty f_2' g_1 + f_1' g_2 \, d\eta = \tfrac{5}{3} g_2'(0). \quad (42)$$

The present analysis can easily be extended to higher order effects. The matching of the third order terms in outer expansions (7) suggests $\varepsilon_2 = Gr^{-2/5}$. However, the first eigen solution

$$\psi = -\lambda(f_1 - 2\eta f_1'/3)\varepsilon_1^{8/3}/x,$$

$$\phi = \lambda(g_1 + 2\eta g_1'/3)\varepsilon_1^{8/3}/x \quad (43)$$

introduces a term in the inner expansion (9) which, in order of magnitude, lies between second and the third term of each series. Here $\lambda$ is an arbitrary constant representing the uncertainty in the effective location of origin.

The boundary-layer characteristics such as wall shear and heat transfer can also be expressed in terms of asymptotic expansions and similarity variables. The coefficient of skin friction is

$$c_f = 2 f_1''(0)\varepsilon + 2 f_2''(0)\varepsilon^2 + 2\lambda f_1''(0)/3\varepsilon^{8/3} + O(\varepsilon^3) \quad (44)$$

where $\varepsilon = Gr_x^{-1/5}$. The local Nusselt numbers for the vertical plate is

$$Nu_x = -g_2'(0) + O(\varepsilon). \quad (45)$$

The average Nusselt number based on the global heat flux is

$$N_Q = J_1\varepsilon^{-1} + J_2 + O(\varepsilon). \quad (46)$$

For the adiabatic wall case the wall temperature $T_a$ may be expressed of recovery factor $r$

$$r = \varepsilon + g_2(0)\varepsilon^2 + \lambda\varepsilon^{8/3} + O(\varepsilon^3). \quad (47)$$

#### 4. THE EFFECTS OF HORIZONTAL BOUNDARY

We here study the effects of an insulated horizontal boundary introduced at distance $d$ below the leading edge of the convective wall plume (Fig. 1) into the otherwise unbounded domain on the buoyancy induced flow. The convective wall plume is placed at $x \geqslant 0$, $y = 0$ and the horizontal boundary at $x = -d$. As in the unbounded medium case we seek the solutions of the problem for large values of Grashof numbers. The analysis for the leading terms in outer and inner expansions is the same as presented in Section 3(i) and we now consider the second order terms in asymptotic expansions. In the outer region the second order stream function is governed by equation (27) and matching condition (28a). The bounding horizontal wall is a streamline as no fluid is entrained into the plume from the region $x < -d$ and the symmetry of flow about $y = 0$ may be expressed as

$$\Psi_2(x = -d, y) = 0, \; |y| \geq 0$$
$$\Psi_2(x, y = 0) = 0, \quad -d < x < 0. \quad (48)$$

In order to bring out certain implications clearly we divide the study into three cases: (i) $d = 0$, (ii) $d \neq 0$ but finite (iii) $d \to \infty$. The third case is studied in Section 3 and the first two are considered below.

(a) *Bounding plane at the leading edge* (d = 0)

The solution of the second order outer flow problem (27) under boundary conditions (28a) and (48) is

$$\Psi_2 = f_1(\infty)(x^2 + y^2)^{3/10} \times \sin\left[\frac{3}{5}\left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right)\right] \Big/ \sin\frac{3\pi}{10}. \quad (49)$$

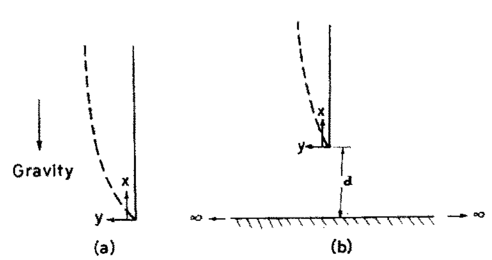


FIG. 1. Convected wall plume: Buoyancy induced flow due to a line thermal source embedded at the leading edge of a vertical plate in (a) unbounded medium (b) medium bounded by an infinite horizontal plane at a distance $x = -d$ from the thermal source.

The matching condition for tangential component of velocity is

$$\Psi_{2y}(x, 0) = \Psi_{2y}(x, \infty) \equiv D_1 x^{-2/5} \quad (50)$$

$$D_1 = -\tfrac{3}{5} f_1(\infty) \cot\left(\frac{3\pi}{10}\right). \quad (51)$$

The second order boundary layer equations satisfying (50) are again the same as (38)–(42) except that the boundary conditions (41a) is replaced by $f_2'(\infty) = D_1$.

(b) *Bounding plane at a finite distance* (d $\neq$ 0)

When the distance between convective wall plume and bounding horizontal plane is finite, the solution to second order outer equation (27) subject to boundary conditions (28a) and (48) is

$$\Psi_2 \sin\frac{3\pi}{5} = f_1(\infty)\left\{(x^2 + y^2)^{3/10} \sin\tfrac{3}{5}\left(\pi - \tan^{-1}\frac{y}{x}\right) - [(x + 2d)^2 + y^2]^{3/10} \sin\left(\frac{3}{5}\tan^{-1}\frac{y}{x + 2d}\right)\right\}. \quad (52)$$

The matching of inner and outer expansions requires that

$$\Psi_{2y}(x, 0) = \psi_{2y}(x, \infty) = A_1 x^{-2/5}\left[1 - \left(\frac{x}{x + 2d}\right)^{2/5} \sec\frac{2\pi}{5}\right]. \quad (53)$$

The displacement speed (53) is zero at $x = x_c$ where $x_c$ is given by

$$x_c = 2d \Big/ \left\{\left[\sec\left(\frac{2\pi}{5}\right)\right]^{5/2} - 1\right\}. \quad (54)$$

The second order boundary layer equations in the inner region in terms of variables

$$\psi_2 = F(x, \eta), \; \phi_2 = x^{-6/5} G(x, \eta), \quad (55)$$

reduce to

$$F''' + \tfrac{3}{5} f_1 F'' + \tfrac{1}{5} f_1' F' + G = x(f_1' F_x' - f_1'' F_x) \quad (56)$$

$$\sigma^{-1} G'' + \tfrac{3}{5}(f_1 G' + 2 f_1' G + F' g_1) = x(f_1' G_x - g_1' F_x) \quad (57)$$

$$F(x, 0) + x F_x(x, 0) = 0, \; F'(x, 0) = 0 \quad (58a, b)$$

$$F'(x, \infty) = A_1\left[1 - \left(\frac{x}{x + 2d}\right)^{2/5} \sec\frac{2\pi}{5}\right] \quad (59)$$

$$G(x, 0) = 0 \;\text{ or }\; G'(x, 0) = 0, \; G(x, \infty) = 0 \quad (60a, b)$$

$$J_2 = \sigma \int_0^\infty F' g_1 + f_1' G \, d\eta. \quad (61)$$

The solutions to the non-similar equations are carried out in terms of two coordinate expansions valid for small and large values of $x$.

For small values of $x$ the expression (59) may be expanded in a series

$$F'(x,\infty) = A_1 + \sum_{m=0}^{\infty} B_m(x/d)^{m+2/5},\ d \neq 0 \quad (62)$$

$$B_m = -A_1 \sec\left(\frac{2\pi}{5}\right)\frac{1}{2^{m+2/5}}\frac{\Gamma(3/5)}{\Gamma(1+m)\Gamma\left(\frac{3-5m}{5}\right)}. \quad (63)$$

The corresponding expansions for $F$ and $G$ are

$$F = f_2 + \sum_{m=0}^{\infty} F_m(\eta)(x/d)^{m+2/5}$$

$$G = g_2 + \sum_{m=0}^{\infty} G_m(\eta)(x/a)^{m+2/5} \quad (64)$$

$$J = J_2 + \sum_{m=0}^{\infty} J_{2m}(\eta)(x/a)^{m+2/5}.$$

The problem governing $(f_2, g_2)$ is the same as described in Section 3(ii) while the problems for $(F_m, G_m)$ are as given below

$$F_m''' + \tfrac{3}{5}f_1F_m'' - \frac{1+5m}{5}f_1'F_m' + \frac{5m+2}{5}f_1''F_m + G_m = 0 \quad (65)$$

$$\sigma^{-1}G_m'' + \tfrac{3}{5}f_1G_m' + \frac{4-5m}{5}f_1'G_m + \frac{5m+2}{5}g_1'F_m + \tfrac{3}{5}g_1F_m' = 0 \quad (66)$$

$$F_m(0) = F_m'(0) = 0,\ G_m(0) \quad \text{or} \quad G_m'(O) = 0 \quad (67)$$

$$F_m'(\infty) = B_m,\ G_m(\infty) = 0. \quad (68)$$

The expression for $J_{2m}$ using the above equations $(m \geq 0)$ is

$$J_{2m} = \sigma\int_0^{\infty} F_m'g_1 + f_1'G_m\,d\eta = -\frac{5}{5m-1}G_m'(0). \quad (69)$$

For large values of $x$ the expression (54) may be expanded in inverse powers of $x$, which after some simplifications may be written as

$$F'(x,\infty) = D_1 + O(1/x). \quad (70)$$

Thus to the lowest order $F$ and $G$ may be expanded as

$$F = \mathscr{F}_2 + O(x^{-1}),\ G = \mathscr{G}_2 + O(x^{-1}). \quad (71)$$

The equations governing $(\mathscr{F}_2, \mathscr{G}_2)$ are identical to those given in Section (4a).

## 5. RESULTS AND DISCUSSION

The equations obtained, in Sections 3 and 4, for the first and second order boundary layers for the vertical convective wall plume in an unbounded fluid and bounded by horizontal plane at an arbitrary distance have been integrated numerically by fourth order Runge–Kutta method with Gill improvement for two values of Prandtl numbers $\sigma = 0.72$ for air and 6.7 for

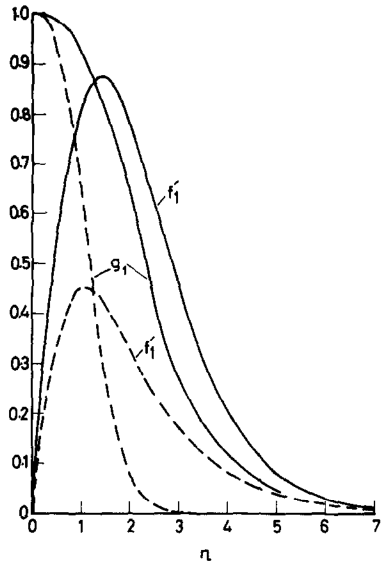


FIG. 2. First order solutions: velocity $f_1'$ and temperature $g_1$ distributions for air and water: —— $\sigma = 0.72$, - - - - - $\sigma = 6.7$.

water. The results for the first order velocity profile $f_1'$ and temperature profile $g_1$, shown in Fig. 2, compare favourably with [2]. The reference temperature $T_r$ is given by

$$T_r = Q_s/KJ_1,\ J_1 = \begin{bmatrix}1.0915\\2.6446\end{bmatrix}. \quad (72)$$

In the square bracket the upper value corresponds to air ($\sigma = 0.72$) and the lower one to water ($\sigma = 6.7$).

For the second order effects we first present the numerical results for the prescribed wall temperature case. For an unbounded medium the velocity and temperature profiles $f_2'$ and $g_2$ are shown in Figs. 3

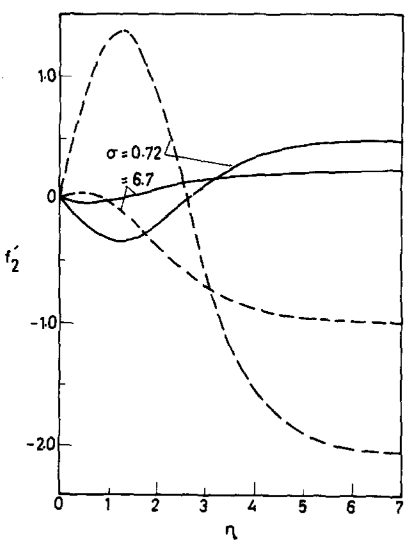


FIG. 3. Second order solutions when wall temperature is prescribed. Velocity $f_2'$ distributions for air and water: —— unbounded medium ($d = \infty$), - - - - - bounded medium when $d = 0$.

and 4 respectively. Coefficient of skin friction local and global heat transfer at any section are

$$c_f = \begin{bmatrix} 2.6201 \\ 1.8596 \end{bmatrix}\varepsilon - \begin{bmatrix} 0.7946 \\ 0.0698 \end{bmatrix}\varepsilon^2 + \lambda\begin{bmatrix} 0.8734 \\ 0.6199 \end{bmatrix}\varepsilon^{8/3} + O(\varepsilon^3) \quad (73)$$

$$Nu_x = \begin{bmatrix} 0.4814 \\ 0.2436 \end{bmatrix} + O(\varepsilon) \quad (74)$$

$$N_Q = \begin{bmatrix} 1.0915 \\ 2.6446 \end{bmatrix}\varepsilon^{-1} - \begin{bmatrix} 0.8022 \\ 0.4060 \end{bmatrix} + O(\varepsilon). \quad (75)$$

The sign of second order contributions to $c_f$ and $N_Q$ are opposite to that of first order. Thus the total skin friction decreases and the thickness of the buoyancy layer increases. This is due to the fact that the displacement speed (30) is positive as the outer inviscid fluid is entrained in the buoyancy layer at an angle $2\pi/5$ with the $x$-axis. The second order effects also decreases the global heat transfer $N_Q$ implying that a part of the heat input from the source of heat is absorbed by the vertical wall. This is also shown by the expression (64) for the heat transfer from the plate.

When the region is bounded by a horizontal plane placed at the leading edge of convected wall plume ($d = 0$) the displacement speed (44) is negative as the fluid enters the buoyancy layer at an angle $7\pi/10$ with the $x$-axis. The velocity and temperature profiles for this case are also displayed in Figs. 3 and 4, respectively, which show that they are of opposite sign

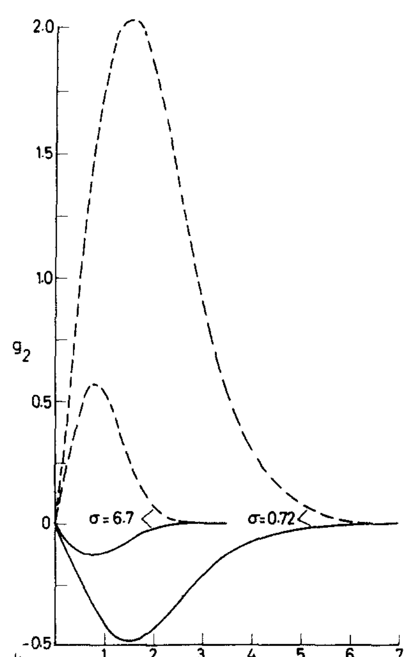


FIG. 4. Second order solution when wall temperature is prescribed. Temperature $g_2$ distributions for air and water: —— unbounded medium ($d = \infty$), ----- bounded medium when $d = 0$.

when compared with the unbounded medium case ($a \to \infty$), implying that second order wall shear and heat transfer should also be of opposite sign with respect to the unbounded case. The buoyancy layer characteristics for the fully bounded region ($a = 0$) case for $\sigma = 0.72$ and 6.7 are

$$c_f = \begin{bmatrix} 2.6201 \\ 1.8596 \end{bmatrix}\varepsilon + \begin{bmatrix} 3.3662 \\ 0.2954 \end{bmatrix}\varepsilon^2 + \lambda\begin{bmatrix} 0.8734 \\ 0.6199 \end{bmatrix}\varepsilon^{8/3} + O(\varepsilon^3) \quad (76)$$

$$Nu_x = -\begin{bmatrix} 2.0391 \\ 1.0334 \end{bmatrix} + O(\varepsilon) \quad (77)$$

$$N_Q = \begin{bmatrix} 1.0915 \\ 2.6446 \end{bmatrix}\varepsilon^{-1} + \begin{bmatrix} 3.3994 \\ 1.6524 \end{bmatrix} + O(\varepsilon). \quad (78)$$

The above results show that for bounded medium cases ($d = 0$) the total skin friction and global heat transfer increase at moderately large values of Grashof numbers. The increase in global heat transfer implies that heat is being transferred from the vertical wall to the buoyancy layer. This is also supported by the expression (77) for heat transfer from the plate.

In the general case when the medium is bounded by a horizontal plane at $x = -d$ the nature of the flow changes depending upon $x$. For $x < x_c$ the displacement speed is positive, the total skin friction and global heat transfer decrease. At $x = x_c$ the displacement speed is zero and the second order contributions vanish. For $x > x_c$ the displacement speed is negative, skin friction and global heat transfer increase at moderately large values of Grashof numbers. The boundary layer characteristics for small values $x$ are ($d \neq 0$)

$$c_f = 2f_1''(O)\varepsilon + \left[2f_2''(O) + \sum_{m=0}^{\infty} 2F_m''(O)(x/d)^{m+2/5}\right] \times \varepsilon^2 + 2\lambda f_1''(O)/3\varepsilon^{8/3} + O(\varepsilon^3) \quad (79)$$

$$Nu_x = -g_2'(O) - \sum_{m=0}^{\infty} G_m'(O)(x/d)^{m+2/5} + O(\varepsilon) \quad (80)$$

$$N_Q = J_1\varepsilon^{-1} + J_2 + \sum_{m=0}^{\infty} J_{2m}(x/d)^{m+2/5} + O(\varepsilon). \quad (81)$$

The solutions to the six terms are given in Table 1. For large values of $x$ the results for the leading terms are the same as given by equations (76) to (78).

We now present our results for an adiabatic convective wall plume. The global heat flux at any section is equal to the heat released $Q_s$ by the convective wall plume at the leading edge. The second order velocity, $f_2'$, and temperature, $g_2$, profiles for the unbounded medium are shown in Figs. 5 and 6 for $\sigma = 0.72$ and 6.7. The coefficient of wall shear and recovery factor are given by

Table 1. Second order solutions for a convective wall plume bounded by a horizontal plane at a finite distance below the leading edge governed by equations (65)–(68)

| | Presented wall temperature | | | | Adiabatic wall | |
|---|---|---|---|---|---|---|
| $m$ | $B_m$ | $F''_m(0)$ | $G'_m(0)$ | $J_m$ | $F''_m(0)$ | $G_m(0)$ |
| | | | $\sigma = 0.72$ | | | |
| 0 | − 1.1898 | − 0.3817 | − 0.1798 | − 1.2488 | 0.4804 | 1.0830 |
| 1 | 0.2379 | 0.2086–2 | − 0.7750–2 | 0.1395–1 | − 0.9608–1 | − 0.2166 |
| 2 | − 0.8329–1 | 0.1699–4 | 0.1321–2 | − 0.1019–2 | 0.3363–1 | 0.7581–1 |
| 3 | 0.3332–1 | − 0.2549–4 | − 0.2567–3 | 0.1273–3 | − 0.1345–1 | − 0.3032–1 |
| 4 | − 0.1416–1 | 0.7978–5 | 0.5595–4 | − 0.2046–4 | 0.5717–2 | 0.1289–1 |
| 5 | 0.6320–2 | − 0.2239–5 | − 0.1327–4 | 0.3849–5 | − 0.2515–2 | − 0.5671–2 |
| 6 | − 0.2803–2 | 0.6424–6 | 0.3358–5 | − 0.8056–5 | 0.1132–2 | 0.2552–2 |
| 7 | 0.1281–2 | − 0.1768–6 | − 0.8938–6 | 0.1832–6 | − 0.5174–3 | − 0.1167–2 |
| | | | $\sigma = 6.7$ | | | |
| 0 | − 0.5769 | − 0.1581 | − 0.1828 | − 0.1364 | 0.6819–1 | 0.4268 |
| 1 | 0.1154 | 0.3514–2 | − 0.3296–2 | 0.6165–3 | − 0.1364–1 | − 0.8537–1 |
| 2 | − 0.4038–1 | − 0.5374–3 | 0.4234–3 | − 0.3551–4 | 0.4774–2 | 0.2988–1 |
| 3 | 0.1615–1 | 0.1108–3 | − 0.6844–4 | 0.3793–5 | − 0.1909–2 | − 0.1195–1 |
| 4 | − 0.6865–2 | − 0.2626–4 | 0.1309–4 | − 0.5726–6 | 0.8115–3 | 0.5079–2 |
| 5 | 0.3024–2 | 0.6787–5 | − 0.2819–5 | 0.1125–6 | − 0.3571–3 | − 0.2235–2 |
| 6 | − 0.1359–2 | − 0.1865–5 | 0.6623–6 | − 0.2792–7 | 0.1607–3 | 0.1006–2 |

$$c_f = \begin{bmatrix} 2.6201 \\ 1.8596 \end{bmatrix} \varepsilon - \begin{bmatrix} 0.3918 \\ 0.0561 \end{bmatrix} \varepsilon^2 + \lambda \begin{bmatrix} 0.8734 \\ 0.6199 \end{bmatrix} \varepsilon^{8/3} + O(\varepsilon^3) \quad (82)$$

$$r = \varepsilon + \begin{bmatrix} 0.4416 \\ 0.1740 \end{bmatrix} \varepsilon^2 + \lambda \varepsilon^{8/3} + O(\varepsilon^3). \quad (83)$$

The second order results show that for an adiabatic convective wall plume the total skin friction and recovery factor decrease whereas the thickness of the buoyancy layer increases at moderately large values of Grashof numbers.

For the bounded medium ($d = 0$) the profile of velocity $f'_2$ and temperature $g_2$ are also shown in Figs. 5 and 6. The corresponding results for skin friction and recovery factor are

$$c_f = \begin{bmatrix} 2.6201 \\ 1.8596 \end{bmatrix} \varepsilon + \begin{bmatrix} 1.6595 \\ 0.2355 \end{bmatrix} \varepsilon^2 + \lambda \begin{bmatrix} 0.8734 \\ 0.6199 \end{bmatrix} \varepsilon^{8/3} + O(\varepsilon^3) \quad (84)$$

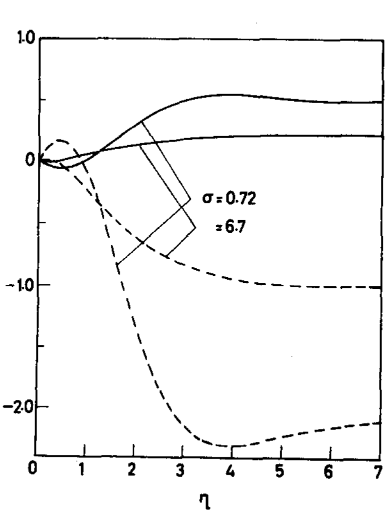


FIG. 5. Second order solution when vertical wall is adiabatic. Velocity $f'_2$ distribution for air and water: —— unbounded medium ($d = \infty$), - - - - - bounded medium ($d = 0$).

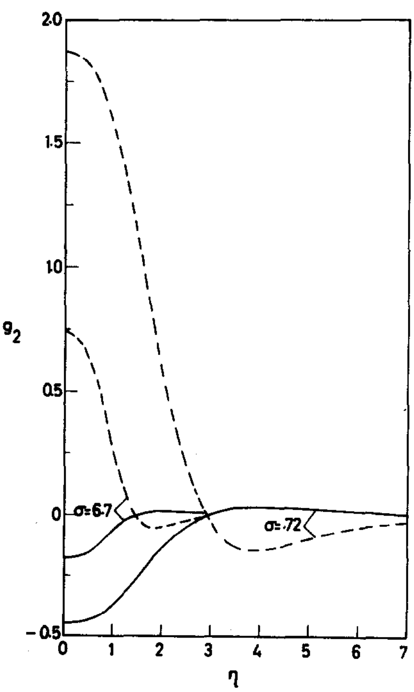


FIG. 6. Second order solutions when vertical wall is adiabatic. Temperature $g_2$ distributions for air and water: —— unbounded medium ($d = \infty$), - - - - - bounded medium ($d = 0$).

$$r = \varepsilon + \begin{bmatrix} 1.8706 \\ 0.7372 \end{bmatrix} \varepsilon^2 + \lambda\varepsilon^{8/3} + O(\varepsilon^2). \quad (85)$$

These results show that for bounded medium ($d = 0$) the total skin friction and recovery factor increases and the thickness of buoyancy layer decreases for moderately large values of Grashof numbers.

In the general case ($d \neq 0$) where the medium is partially bounded by a horizontal plane at $x = -d$, the skin friction and recovery factors for small values of $x$ are given by

$$c_f = 2f_1''(O) + 2\left[f_2''(O) + \sum_{m=0}^{\infty} F_m''(O)(x/d)^{m+2/5}\right] \times \varepsilon^2 + 2\lambda f_1''(O)/3\varepsilon^{8/3} + O(\varepsilon^3) \quad (86)$$

$$r = \varepsilon + \left[g_2(O) + \sum_{m=0}^{\infty} G_m(O)(x/d)^{m+2/5}\right] \times \varepsilon^2 + \lambda\varepsilon^{8/3} + O(\varepsilon^3). \quad (87)$$

The various coefficients obtained from the solutions of the equations are also given in Table 1. For large values of $x$ the solution to the leading terms in the expansions is given by equations (84) and (85). Further, from earlier discussion it follows that the skin friction and recovery factor decrease for $x < x_c$, increases for $x > x_c$ and for $x = x_c$ the second order contribution vanishes.

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## APPENDIX: GLOBAL HEAT FLUX CONSIDERATIONS

The first order boundary-layer equations are not correct for $x \sim O(1)$ and heat transfer has an error of order $\varepsilon_1^{-5/3}$ when $x \sim O(\varepsilon_1^{5/3})$, and by integrating heat conduction we lose a term in heat transfer which represents the concentrated heat source at the leading edge. From an order of magnitude analysis near the leading edge we consider the following variables:

$$x = \varepsilon_1^{5/3}\tilde{x},\ y = \varepsilon_1^{5/3}\tilde{y},\ \psi = \varepsilon_1^2\tilde{\psi},\ \phi = \tilde{\phi}. \quad (A1)$$

The global heat flux from the vertical plate can be estimated by integrating [4] the energy equation expressed in the variables (A1) along a large control surface $\Omega$ to get

$$q + \int_{\Omega} \nabla\phi \cdot \hat{n}d\tilde{s} = \sigma\int_{\Omega} \phi\mathbf{V} \cdot \hat{n}d\tilde{s},\ \mathbf{V} = (\tilde{\psi}_{\tilde{y}}, -\tilde{\psi}_{\tilde{x}}) \quad (A2)$$

where $q$ is the nondimensional heat input from the source per unit length. For a large rectangle control surface $\Omega$ with sides $y = 0$, $y = 1$, $x = -1$ and $x = x_0$, where $x_0 \sim O(1)$, the expression (A2) may be simplified and expressing the tilde variables in favour of untilde variables from (A1) the result for global heat flux $Q$ can be written as

$$Q \equiv \varepsilon_1 qKT_r - \int_0^{x} KT_y(x, O)dx = KT_rJ \quad (A3)$$

$$J \equiv \left[\sigma\int_0^{\infty} \psi_y\phi dy - \varepsilon_1^2\int_0^{\infty} \phi_x dy + O(\varepsilon_1^2)\right] \Big/ \varepsilon_1. \quad (A4)$$

In global heat flux expression (A3) the first term side is the heat input $Q_s$ ($= \varepsilon_1 qKT_r$) from the line source embedded at the leading edge and the second term the integrated conductive heat transfer from the vertical wall.

## PANACHE CONVECTIF EN PAROI: ANALYSE D'ORDRE ELEVE


**Résumé** — On étudie, par la méthode des développements asymptotiques, la convection naturelle issue d'une source thermique, fixe, linéaire et noyée au bord d'attaque d'une surface verticale, pour des valeurs modérément élevées du nombre de Grashof. Les équations de couche limite de premier et de second ordre sont traitées pour le milieu limité par un plan infini et horizontal, placé à distance au dessous du panache de paroi. Les solutions numériques sont obtenues pour deux fluides, l'air et l'eau. On montre que la structure du panache dépend fortement de la situation du plan horizontal. On présente séparément des résultats pour une température de paroi donnée et pour un panache adiabatique.


## KONVEKTIVE AUFTRIEBSSTRÖMUNG AN EINER WAND: EINE ANALYSE HÖHERER ORDNUNG


**Zusammenfassung**—Für mäßig große Grashof-Zahlen wird die natürliche Konvektionsströmung, die von einer stationären linienförmigen Wärmequelle ausgeht, nach der Methode der angepaßten asymptotischen Entwicklung untersucht. Die Wärmequelle sitzt dabei an der Vorderkante einer senkrechten Platte. Diskutiert werden die Grenzschichtgleichungen 1. und 2. Ordnung für den Fall, daß das Medium nach unten durch eine unendlich große horizontale Fläche begrenzt wird. Diese Fläche wird in beliebigem Abstand unterhalb der konvektiven Wandanlaufströmung angebracht. Numerische Lösungen werden für die beiden Medien Luft und Wasser erhalten. Es zeigt sich, daß die Struktur der erzwungenen Auftriebsströmung sehr stark von der Lage der waagerechten Ebene abhängig ist. Gesonderte Ergebnisse werden für den Fall vorgegebener Wandtemperatur als auch für eine Auftriebsströmung bei adiabater Wand dargestellt.

# СВОБОДНОКОНВЕКТИВНАЯ ВОСХОДЯЩАЯ СТРУЯ НА СТЕНКЕ. АНАЛИЗ С ПОМОЩЬЮ СТАРШИХ ЧЛЕНОВ РАЗЛОЖЕНИЯ

**Аннотация** — Методом сращиваемых асимптотических разложений проведено исследование свободноконвективного течения, возникающего от стационарного линейного источника тепла, заделанного в переднюю кромку вертикальной поверхности, при умеренно больших значениях числа Грасгофа. Анализируются уравнения пограничного слоя первого и второго порядка в случае, когда внизу под конвективной струей на произвольном расстоянии помещается неограниченная горизонтальная поверхность. Получены численные решения для двух жидкостей, а именно: для воздуха и воды. Показано, что на структуру конвективной струи на стенке сильное влияние оказывает положение горизонтальной плоскости. Приведены некоторые результаты для случая заданной температуры стенки и адиабатической конвективной струи на стенке.